On two open points in Part IV of Ramanujan's Lost Notebook: the domain of Entry 12.2.3 and a summation reading of Entry 19.2.2

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Abstract

We discuss two places in Andrews and Berndt, *Ramanujan's Lost Notebook, Part IV*, where a question is left open. (1) For Entry 12.2.3, $L(s,\chi)L(s-r,\chi) = \sum \chi(n)\sigma_r(n)n^{-s}$ with χ the non-principal character modulo 4, the authors expect the identity to hold for $\operatorname{Re} s > \sup\{0, \operatorname{Re} r\}$. We show that this fails for $r = 0$: the abscissa of convergence of $\sum \chi(n)d(n)n^{-s}$ lies in $[\frac{1}{4}, \frac{1}{3}]$, so the series diverges for $0 < \operatorname{Re} s < \frac{1}{4}$ although both series on the left converge there. The proof combines Kronecker's lemma with a mean-square theorem of Cao, Tanigawa and Zhai and an upper bound of de Roton. (2) For Entry 19.2.2, which the authors call false because it contains divergent series, we show that for $s < -2$ and quadratic irrational $\alpha/2\pi$ the series $\sum \sigma_s(n)\sin(n\alpha)$ is Abel summable, with value $\frac{1}{2}\sum_d d^s \cot(d\alpha/2)$, but that the identity still fails with this reading; numerically the value has poles at the rational points $\alpha/2\pi = p/q$, which suggests why an identification with the smooth double integral in (19.4.2) was not found. A verification script accompanies the note.

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1 Introduction

Chapter 12 of [2] records a manuscript of Ramanujan on infinite series. With χ the non-principal character modulo 4 and $\sigma_r(n) = \sum_{d|n} d^r$, Entry 12.2.3 (p. 318 of [1]) states that for $\operatorname{Re} s > 1$, $\operatorname{Re}(s-r) > 1$,

$$\sum_{m\geq 1} \frac{\chi(m)}{m^s} \sum_{n\geq 1} \frac{\chi(n)}{n^{s-r}} = \sum_{n\geq 1} \frac{\chi(n)\sigma_r(n)}{n^s}. \tag{1}$$

After the (easy) proof the authors write: "We expect that the domain of validity can be extended to $\operatorname{Re} s > \sup\{0, \operatorname{Re} r\}$, but we are unable to prove this" [2, p. 270]. This is the region where both series on the left converge. In Section 2 we show that the expectation fails already for $r = 0$.

Chapter 19 of [2] begins with two formulas from page 336 of [1] that are "clearly wrong" because they contain divergent series. The second, Entry 19.2.2, asserts that for $\alpha, \beta > 0$ with $\alpha\beta = 4\pi^2$,

$$\alpha^{(s+1)/2}\Big\{\tfrac{1}{\alpha}\zeta(1-s) + \tfrac{1}{2}\zeta(-s)\tan\tfrac{\pi s}{2} + \sum_{n\geq 1}\sigma_s(n)\sin(n\alpha)\Big\}$$
$$= \beta^{(s+1)/2}\Big\{\tfrac{1}{\beta}\zeta(1-s) + \tfrac{1}{2}\zeta(-s)\tan\tfrac{\pi s}{2} + \sum_{n\geq 1}\sigma_s(n)\sin(n\beta)\Big\}. \tag{2}$$

Andrews and Berndt compare it with Ramanujan's correct formula (19.4.1), which has a double integral in place of the series, and write that they "are unable to make any identification of the double integrals in (19.4.2) with the divergent sums in (19.2.2)" [2, p. 380]. In Section 3 we test the most natural reading of the divergent series, by Abel summation.

We do not claim new methods; both points are applications of standard tools. We found no discussion of either point in the literature citing [2] (zbMATH Open, OpenAlex, September 2026).

2 Entry 12.2.3 with $r = 0$

Let $A(x) = \sum_{n\le x} \chi(n)d(n)$ and let σ_c be the abscissa of convergence of $F(s) = \sum_{n\ge 1} \chi(n)d(n)n^{-s}$. For $\operatorname{Re} s > 1$, $F(s) = L(s,\chi)^2$.

Theorem 1. $\frac{1}{4} \le \sigma_c \le \frac{1}{3}$. *Consequently $F(s)$ diverges for every s with $\operatorname{Re} s < \frac{1}{4}$, and (1) with $r = 0$ holds for $\operatorname{Re} s > \frac{1}{3}$ but not for $0 < \operatorname{Re} s < \frac{1}{4}$, although $\sum \chi(m)m^{-s}$ converges for $\operatorname{Re} s > 0$.*

The lower bound is a consequence of classical Ω-results for Dirichlet series with a functional equation of this type (Landau [6]; Chandrasekharan and Narasimhan [7]); we derive it from a recent mean-square theorem whose hypotheses can be checked directly. We use the class $S^{\theta}_{\mathrm{real}}$ of Cao, Tanigawa and Zhai [3]: Dirichlet series $L(s) = \sum a(n)n^{-s}$ with real coefficients such that $(s-1)^m L(s)$ is entire of finite order for some $m \ge 0$; $\Lambda(s) = L(s)Q^s \prod_j \Gamma(\alpha_j s + \beta_j)$ satisfies $\Lambda(s) = \omega\Lambda(1-\bar{s})$ with $Q > 0$, $|\omega| = 1$, $\alpha_j > 0$, $\operatorname{Re}\beta_j \ge 0$, $\sum \beta_j$ real; and $|a(n)| \ll n^{\theta+\varepsilon}$, $\sum_{n\le y} |a(n)|^2 \ll y^{1+\varepsilon}$. The degree is $d = 2\sum \alpha_j$. The error term is $E(y) = A^*(y) - Q(y)$, where $A^*(y) = \sum_{n\le y}{}' a(n)$ (last term halved when y is an integer) and $Q(y)$ is the sum of the residues of $L(s)y^s/s$.

Lemma 2. $L(s,\chi)^2 \in S^0_{\mathrm{real}}$*, of degree* 2*, and* $Q(y) = \frac{1}{4}$.

Proof. $L(s,\chi)$ is entire of finite order, and since χ is odd and primitive of conductor 4 with root number 1, $\xi(s) = (4/\pi)^{s/2}\Gamma(\frac{s+1}{2})L(s,\chi)$ satisfies $\xi(s) = \xi(1-s)$. Hence $\Lambda(s) = (4/\pi)^s\Gamma(\frac{s+1}{2})^2 L(s,\chi)^2$ satisfies $\Lambda(s) = \Lambda(1-s)$, with $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \frac{1}{2}$, $Q = 4/\pi$, $\omega = 1$, $d = 2$. The coefficients $\chi(n)d(n)$ are real, $|\chi(n)d(n)| \le d(n) \ll n^{\varepsilon}$ and $\sum_{n\le y} d(n)^2 \ll y\log^3 y$. The only pole of $L(s,\chi)^2 y^s/s$ is at $s = 0$, with residue $L(0,\chi)^2 = \frac{1}{4}$. ☐

Proof of Theorem 1. By [3, Corollary 1] (degree 2, $\theta = 0 \le \frac{1}{4}$) and Lemma 2,

$$\int_1^T E(y)^2\, dy = C_2 T^{3/2} + O(T^{1+\varepsilon}), \qquad C_2 > 0. \tag{3}$$

Suppose $\sigma_c < \frac{1}{4}$ and choose a real s_0 with $\max\{\sigma_c, 0\} < s_0 < \frac{1}{4}$. Then $\sum \chi(n)d(n)n^{-s_0}$ converges, so $A(x) = o(x^{s_0})$ by Kronecker's lemma (partial summation), hence $E(y) = o(y^{s_0})$ and $\int_1^T E^2 = o(T^{1+2s_0}) = o(T^{3/2})$, contradicting (3). Thus $\sigma_c \ge \frac{1}{4}$; a Dirichlet series that converges at s_1 converges for $\operatorname{Re} s > \operatorname{Re} s_1$, so F diverges whenever $\operatorname{Re} s < \frac{1}{4}$. For the upper bound, $L(s,\chi)^2$ lies in the extended Selberg class, so de Roton's estimate [4] (as stated in [3, §1.1]) gives $E(y) \ll y^{(d-1)/(d+1)+\varepsilon} = y^{1/3+\varepsilon}$; partial summation then gives convergence of $F(s)$ for $\operatorname{Re} s > \frac{1}{3}$, and $F(s) = L(s,\chi)^2$ there by analytic continuation. ☐

Remark 3. For real r the same heuristic, applied to $L(s+\frac{r}{2},\chi)L(s-\frac{r}{2},\chi) = \sum \chi(n)\sigma_r(n)n^{-r/2}\, n^{-s}$, predicts divergence of the right side of (1) for $\sup\{0, r\} < s < \frac{1}{4} + \frac{r}{2}$, an interval that is non-empty exactly when $|r| < \frac{1}{2}$. Numerically, with $A_r(x) = \sum_{n\le x} \chi(n)\sigma_r(n)$, the mean of $A_r(x)^2/x^{1/2+r}$ over

$[X/2, X]$ is of order 1 for $r = \pm\frac{1}{4}$ and $X = 10^5, 10^6$ (check A6 of the script). We do not claim this case: the coefficients $\chi(n)\sigma_r(n)n^{-r/2}$ do not satisfy $\sum_{n\le y}|a(n)|^2 \ll y^{1+\varepsilon}$ when $r \neq 0$, so [3, Corollary 1] does not apply as stated.

Table 1 illustrates (3); $A(x)$ is computed exactly by the hyperbola method

$$A(x) = 2\sum_{a\le\sqrt{x}}\chi(a)S(x/a) - S(\sqrt{x})^2, \qquad S(m) = \sum_{b\le m}\chi(b).$$

The partial sums of $\sum\chi(n)d(n)n^{-1/5}$ oscillate with range 8.970, 10.795, 14.929 on $[10^k, 10^{k+1}]$, $k = 3, 4, 5$ (Figure 1, left).

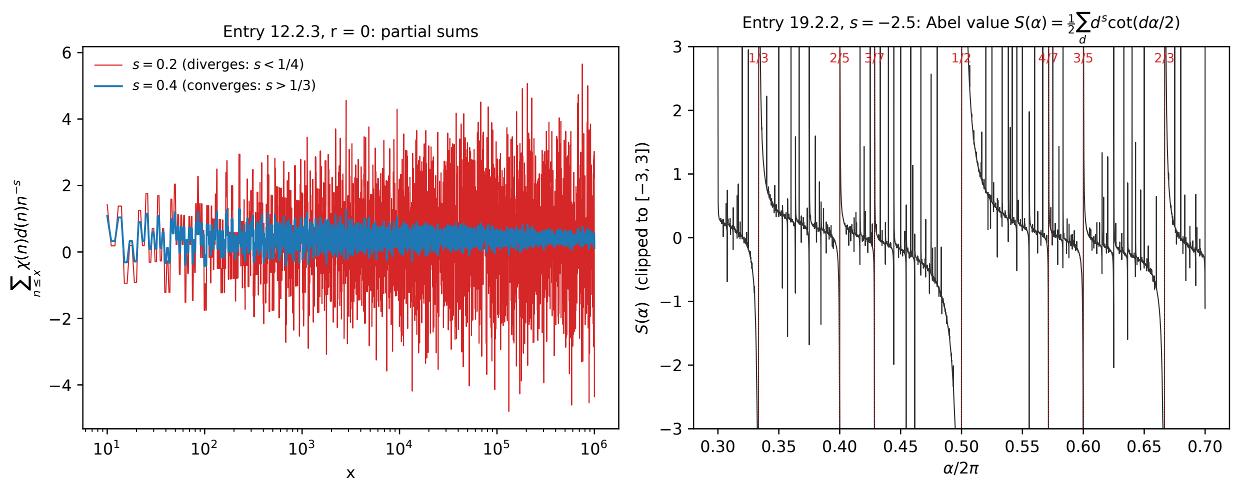

Figure 1: Left: partial sums of $\sum\chi(n)d(n)n^{-s}$ for $s = 0.2$ (divergent, Theorem 1) and $s = 0.4$ (convergent). Right: the Abel value $S(\alpha)$ of Section 3 for $s = -2.5$, with poles at rational $\alpha/2\pi$.

X	10^6	10^8	10^{10}	10^{12}
mean of $A(x)^2/\sqrt{x}$ over 600 random $x \in [X/2, X]$	0.7322	0.7815	0.7571	0.8151

Table 1: The mean square of $A(x)$ is of order $x^{1/2}$, as in (3).

3 Entry 19.2.2 read by Abel summation

Put $\xi = \alpha/2\pi$ and $S(\alpha) = \frac{1}{2}\sum_{d\ge1} d^s\cot(d\alpha/2)$.

Proposition 4. *Let $s < -2$ and let ξ be irrational with $\|d\xi\| \ge c/d$ for all $d \ge 1$ and some $c > 0$ (for example, ξ a quadratic irrational). Then $S(\alpha)$ converges absolutely and $\sum_{n\ge1}\sigma_s(n)\sin(n\alpha)$ is Abel summable to $S(\alpha)$.*

Proof. For $0 < t < 1$ and $\beta \notin 2\pi\mathbb{Z}$ put $f_t(\beta) = \sum_{m\ge1} t^m\sin(m\beta) = t\sin\beta/(1 - 2t\cos\beta + t^2)$. Since $1 - 2t\cos\beta + t^2 \ge 4t\sin^2(\beta/2)$, we have $|f_t(\beta)| \le 1/(2|\sin(\beta/2)|)$, and $f_t(\beta) \to \frac{1}{2}\cot(\beta/2)$ as $t \to 1^-$. With $\beta = d\alpha$, $|\sin(\pi d\xi)| \ge 2\|d\xi\| \ge 2c/d$, so $|d^s f_t(d\alpha)| \le d^{s+1}/(4c)$, which is summable for $s < -2$. As σ_s is bounded, $\sum_n \sigma_s(n)\sin(n\alpha)t^n = \sum_d d^s f_{t^d}(d\alpha)$ for $0 < t < 1$, and dominated convergence gives the limit $S(\alpha)$ as $t \to 1^-$. $\square$

Since Cesàro $(C, 1)$ summability implies Abel summability with the same value [5], the $(C, 1)$ reading of (2), where it exists, gives the same numbers. Let $\mathcal{L}(\alpha)$ denote the left side of (2) with the series replaced by $S(\alpha)$.

Corollary 5. *Entry 19.2.2 is false also in the Abel reading. For $s = -2.5$ and $\alpha = 2\pi\sqrt{2}$, $\beta = 4\pi^2/\alpha$, $\mathcal{L}(\alpha) - \mathcal{L}(\beta) = 0.1408\ldots$; similarly 0.0658 and -0.0464 for $\xi = \frac{1+\sqrt{5}}{2}$ and $\sqrt{3}$, and 0.0422, 0.0305, 0.0211 for $s = -4.5$.*

Proof. ξ and $1/\xi$ are quadratic irrationals, so Proposition 4 applies to both α and β. If the partial quotients of ξ are bounded by M, then $\|d\xi\| > 1/((M+2)d)$ for all $d \geq 1$ [8]; here $M \leq 2$, so $c = \frac{1}{4}$ is admissible for ξ and $1/\xi$. Truncating S at $d \leq D = 4 \cdot 10^6$, the tail is at most $\sum_{d>D} d^{s+1}/(4c) \leq D^{s+2}/(-s-2)$. The total truncation error is below 10^{-3} for $s = -2.5$ and below 10^{-17} for $s = -4.5$, far smaller than the differences stated. □

Remark 6. The terms with $q \mid d$ suggest that S has a pole at each $\alpha_0 = 2\pi p/q$ with residue $q^{s-1}\zeta(1-s)$. Numerically, for $s = -2.5$ and $\alpha = \alpha_0 + 10^{-6}$, $(\alpha - \alpha_0)S(\alpha)$ agrees with $q^{s-1}\zeta(1-s)$ (see also Figure 1, right) to relative accuracy $1.4 \cdot 10^{-10}$, $1.0 \cdot 10^{-5}$, $2.0 \cdot 10^{-5}$, $3.3 \cdot 10^{-5}$ for $p/q = \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}$. The double integral in (19.4.1), by contrast, is a smooth function of $\alpha > 0$ for $s > -1$, and its analytic continuation to $-3 < s < -1$ (computed numerically) shows no such singularities. So the divergent series in (2) cannot be matched pointwise with the double integrals under this reading, which may explain why no identification was found.

Data availability. The script `verify_ln4_remarks.py` (10 checks, requiring NumPy and mpmath) reproduces every number in this note, including Table 1; an offline calculator for $A(x)$ and for the two sides of (2), and the code for Figure 1 are available at `https://github.com/selloriybrendi/ramanujan-lost-notebook-iv-remarks`; the calculator runs at `https://selloriybrendi.github.io/ramanujan-lost-notebook-iv-remarks/`.

Use of generative AI

Claude Opus 5.5 (Anthropic, model identifier `claude-opus-5-5`), accessed through Claude Code in September 2026, was used to locate the open questions in [2], to find the mean-square theorem in [3] and check its hypotheses, to carry out the numerical computations reported here (with NumPy and mpmath) under the author's direction, and to help prepare the text. It was used to verify every stated number independently. The author directed the work, checked all results and takes full responsibility for the content.

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