

The constants K and λ in Ramanujan's identity (9.2.5) of the Lost Notebook

Otakhon U. Kenjaev

Independent researcher, Khorezm, Uzbekistan

ORCID 0009-0009-3566-9285

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Abstract

On pages 270–271 of the Lost Notebook, Ramanujan stated an identity for the alternating divisor series $\sum (-1)^{n+1} \sigma_{s-1}(n) e^{-n\pi\sqrt{3}}$ whose right-hand side contains two undefined quantities K and λ . Andrews and Berndt recorded it as (9.2.5) and wrote that they were unable either to identify K, λ or to obtain an identity of Ramanujan's form. We show that the identity holds for every $\operatorname{Re} s > 2$ with $K = \mu - \nu$, $\lambda = \mu + \nu$, the sum running over coprime $\mu, \nu \geq 1$. The proof groups the Andrews–Berndt lattice sum by the six units of the Eisenstein integers. As consequences we obtain a fast evaluation of the hexagonal sector lattice sum $R(s)$ and the closed forms $R(6k) = (-1)^k (c_k E_6(\rho)^k - 3)/3$ for all $k \geq 1$. We also report randomized numerical checks.

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1 Introduction

Let $\sigma_{s-1}(n) = \sum_{d|n} d^{s-1}$ and

$$L(s) = \sum_{n \geq 1} (-1)^{n+1} \sigma_{s-1}(n) e^{-n\pi\sqrt{3}}. \quad (1)$$

In Section 18 of the manuscript printed on pages 270–271 of the Lost Notebook [1], Ramanujan wrote an identity of the form

$$L(s) = -\frac{2\Gamma(s)\zeta(s)}{(2\pi)^s} \left\{ \cos \frac{\pi s}{6} + 2 \cos \frac{\pi(s+1)}{6} \cos \frac{\pi(s-1)}{6} \sum \frac{\cos(s \tan^{-1} \frac{K}{\lambda\sqrt{3}})}{(\mu^2 + \mu\nu + \nu^2)^{s/2}} \right\}, \quad (2)$$

recorded by Andrews and Berndt as (9.2.5) in [2, §9.2]. After (9.3.2) they write that K and λ are undefined and that they are “unable either to identify them or to obtain an identity of the form given by Ramanujan”. They prove instead an identity for $L(s)$ as a sum over the half-plane $\mu \geq 1$, $\nu \in \mathbb{Z}$ (Section 2).

Prior literature. Berndt and Pongsriam [4] observe that pages 270–271 were evidently intended as a final Section 18 of Ramanujan's paper [5]; they state the companion identity (9.2.4) as their Entry 5.1 and refer to [2] for both identities. The commentary on Part IV in [3, §18.4] does not

return to (9.2.5). We found no later identification of K, λ among the works citing [2] (OpenAlex, zbMATH Open and Google Scholar, September 2026).

Contribution. (i) Theorem 1: (2) holds with $K = \mu - \nu$, $\lambda = \mu + \nu$, $\gcd(\mu, \nu) = 1$, $\mu, \nu \geq 1$, for $\operatorname{Re} s > 2$. (ii) Theorem 2: the slowly convergent sum $R(s)$ in (3) is expressed through the rapidly convergent series (1). (iii) Corollary 1: closed forms of $R(6k)$ for all k . We do not claim new methods: at integer s the identity is the orbit decomposition of the classical lattice sum for the Eisenstein series $E_s(\rho)$ (Remark 4). The new point is the identification of Ramanujan's K, λ and its consequences for real and complex s .

2 Notation and the Andrews–Berndt identity

Let $\omega = e^{i\pi/3}$, so $\omega^2 = \omega - 1$, and let $\rho = e^{2\pi i/3}$. For $w = a + b\omega \in \mathbb{Z}[\omega]$ put $N(w) = |w|^2 = a^2 + ab + b^2$; w is *primitive* if $\gcd(a, b) = 1$. The units are $\pm 1, \pm\omega, \pm\omega^2$. Write

$$R(s) = \sum_{\substack{a, b \geq 1 \\ \gcd(a, b) = 1}} \frac{\cos(s\psi_w)}{N(w)^{s/2}}, \quad \psi_w = \arg w - \frac{\pi}{6} \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right), \quad \tan \psi_w = \frac{b - a}{\sqrt{3}(a + b)}. \quad (3)$$

The series converges absolutely for $\operatorname{Re} s > 2$. Andrews and Berndt [2, p. 217] prove that, for $\operatorname{Re} s > 2$,

$$L(s) = -\frac{\Gamma(s)\zeta(s)}{(2\pi)^s} \sum_{\substack{\mu \geq 1, \nu \in \mathbb{Z} \\ \gcd(\mu, \nu) = 1}} \frac{\cos(s \arg z)}{|z|^s}, \quad z = e^{i\pi/6}(\mu + \nu\omega). \quad (4)$$

Here $|z|^2 = \mu^2 + \mu\nu + \nu^2$ and $\tan(\arg z) = (\mu + 2\nu)/(\sqrt{3}\mu)$ with $|\arg z| < \pi/2$.

3 Two lemmas

Lemma 1 (orbit partition). *Let $H = \{\mu + \nu\omega : \mu \geq 1, \nu \in \mathbb{Z}\}$. If w is primitive and not a unit, exactly three of its six associates lie in H , namely $w, \omega^{-1}w, \omega^{-2}w$ when $w = a + b\omega$ with $a, b \geq 1$; moreover every such orbit contains exactly one element with $a, b \geq 1$. For the unit orbit exactly two associates, 1 and ω^{-1} , lie in H , and $\pm\omega$ lie on its boundary.*

Proof. Write $\mu + \nu\omega = x + iy$. Then $\mu = x - y/\sqrt{3}$, so for integers $\mu \geq 1 \iff \mu > 0 \iff \arg(\mu + \nu\omega) \in (-2\pi/3, \pi/3)$, an open sector of width π . Multiplication by ω adds $\pi/3$ to the argument, so the six associates of w have arguments $\theta + j\pi/3$ ($0 \leq j \leq 5$). If $\theta \notin (\pi/3)\mathbb{Z}$, exactly three of them fall in an open sector of width π , and exactly one falls in the open sector $(0, \pi/3)$, which is the set $a, b \geq 1$. If $\theta \in (\pi/3)\mathbb{Z}$, then $w = tu$ with $t > 0$ and u a unit; primitivity forces $t = 1$. The associates of 1 have arguments $0, \pm\pi/3, \pm 2\pi/3, \pi$; the open sector contains 0 and $-\pi/3$ (the elements 1 and ω^{-1}), and $\pm\omega$ lie on its boundary rays $\mu = 0$. Units act by matrices in $\operatorname{GL}_2(\mathbb{Z})$ on (a, b) , hence preserve primitivity. $\square$

Lemma 2 (sign). $\cos(\operatorname{stan}^{-1}(-x)) = \cos(\operatorname{stan}^{-1}x)$. Hence in (2) the choices $K = \mu - \nu$ and $K = \nu - \mu$ give the same sum; the set $\{a, b \geq 1, \gcd = 1\}$ and $N(w)$ are also symmetric under $a \leftrightarrow b$.

4 Main results

Theorem 1. *For $\operatorname{Re} s > 2$, identity (2) holds with $K = \mu - \nu$, $\lambda = \mu + \nu$ and the sum over coprime $\mu, \nu \geq 1$; equivalently*

$$L(s) = -\frac{2\Gamma(s)\zeta(s)}{(2\pi)^s} \left\{ \cos \frac{\pi s}{6} + 2 \cos \frac{\pi(s+1)}{6} \cos \frac{\pi(s-1)}{6} R(s) \right\}.$$

Proof. By absolute convergence we may regroup (4) by unit orbits (Lemma 1). For $w = a + b\omega$ with $a, b \geq 1$ the three associates in H have $\arg z = \psi_w + \pi/3$, ψ_w , $\psi_w - \pi/3$ (since $z = e^{i\pi/6}w$), and

$$\cos(s(\psi + \frac{\pi}{3})) + \cos(s\psi) + \cos(s(\psi - \frac{\pi}{3})) = \cos(s\psi)(1 + 2 \cos \frac{\pi s}{3}) = 4 \cos \frac{\pi(s+1)}{6} \cos \frac{\pi(s-1)}{6} \cos(s\psi).$$

The unit orbit contributes $\cos(\pi s/6) + \cos(-\pi s/6) = 2 \cos(\pi s/6)$. Substituting into (4) gives the claim, and $\tan \psi_w = (b - a)/(\sqrt{3}(a + b))$ with Lemma 2 gives $K = \mu - \nu$, $\lambda = \mu + \nu$ (with $a = \mu$, $b = \nu$). $\square$

Theorem 2 (fast evaluation). *If $\operatorname{Re} s > 2$ and $s \not\equiv 2, 4 \pmod{6}$, then*

$$R(s) = \frac{-\frac{L(s)(2\pi)^s}{2\Gamma(s)\zeta(s)} - \cos \frac{\pi s}{6}}{2 \cos \frac{\pi(s+1)}{6} \cos \frac{\pi(s-1)}{6}}.$$

Since $e^{-\pi\sqrt{3}} \approx 0.0043$, about twenty terms of (1) give double precision, while the lattice sum (3) truncated at $N(w) \leq X^2$ has error of order X^{2-s} .

Remark 1 (degenerate values). For $s \equiv 2, 4 \pmod{6}$ the coefficient of $R(s)$ vanishes and Theorem 1 carries no information on $R(s)$. For even integers s in these classes one has $E_s(\rho) = 0$, because every monomial $E_4^i E_6^j$ of weight s then contains E_4 and $E_4(\rho) = 0$.

Corollary 1 (closed forms). *For every integer $k \geq 1$,*

$$R(6k) = (-1)^k \frac{E_{6k}(\rho) - 3}{3}, \quad E_{6k}(\rho) = c_k E_6(\rho)^k, \quad E_6(\rho) = \frac{27 \Gamma(1/3)^{18}}{512 \pi^{12}},$$

where $c_k \in \mathbb{Q}$ is the coefficient of E_6^k when E_{6k} is written as a polynomial in E_4, E_6 . The first values are $c_1 = 1$, $c_2 = \frac{250}{691}$, $c_3 = \frac{5500}{43867}$, $c_4 = \frac{10285000}{236364091}$, $c_5 = \frac{26021050000}{1723168255201}$ (exact values for $k \leq 12$ are archived with the scripts [8]). In particular

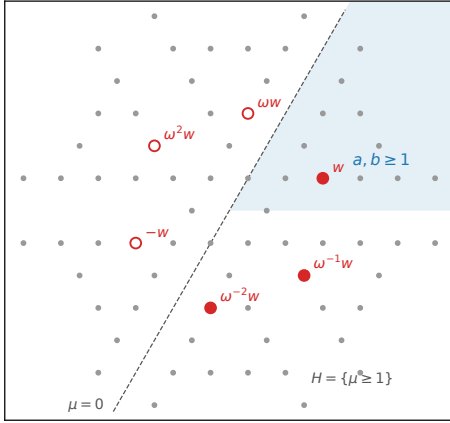
$$R(6) = 1 - \frac{9 \Gamma(1/3)^{18}}{512 \pi^{12}}, \quad R(12) = \frac{30375 \Gamma(1/3)^{36}}{90570752 \pi^{24}} - 1.$$

Proof. With $q = e^{2\pi i \rho} = -e^{-\pi\sqrt{3}}$ we have $L(s) = -\sum \sigma_{s-1}(n)q^n = \frac{B_s}{2s}(E_s(\rho) - 1)$ for even s . At $s = 6k$: $\cos(\pi k) = (-1)^k$, $2 \cos \frac{\pi(6k+1)}{6} \cos \frac{\pi(6k-1)}{6} = \frac{3}{2}$, and $\zeta(6k) = (-1)^{3k+1} B_{6k} (2\pi)^{6k} / (2(6k)!)$, which give $\frac{L(6k)(2\pi)^{6k}}{2\Gamma(6k)\zeta(6k)} = (-1)^{k+1} \frac{E_{6k}(\rho) - 1}{2}$. Theorem 2 then gives the formula for $R(6k)$. The value of $E_6(\rho)$ is the Chowla–Selberg evaluation [7], and $E_4(\rho) = 0$ kills every monomial containing E_4 . $\square$

Remark 2 (consistency). As $k \rightarrow \infty$, $E_{6k}(\rho) \rightarrow 3$ (the six units contribute $u^{-6k} = 1$ each, halved), so $R(6k) \rightarrow 0$, in agreement with $R(s) \sim 3^{-s/2}$ from the smallest term $w = 1 + \omega$. Numerically

$$E_{36}(\rho) = 3.00000000774\dots, \quad E_{60}(\rho) = 3.00000000000001\dots$$

Lemma 1: the orbit of $w = 2 + \omega$ meets H three times (filled)



dotted: $s \equiv 2, 4 \pmod{6}$, where the coefficient of $R(s)$ vanishes

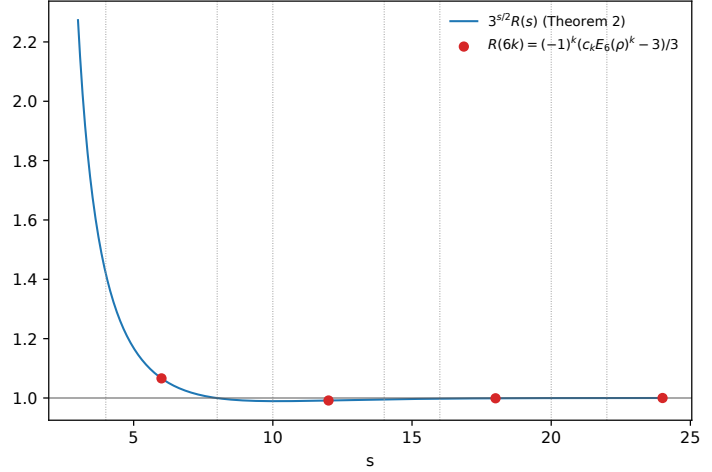


Figure 1: Left: primitive points of $\mathbb{Z}[\omega]$; the orbit of $w = 2 + \omega$ meets the half-plane $H = \{\mu \geq 1\}$ in the three filled points (Lemma 1); shaded: the sector $a, b \geq 1$. Right: $3^{s/2}R(s)$ computed from Theorem 2 (from the lattice sum near the dotted lines $s \equiv 2, 4 \pmod{6}$); dots: Corollary 1.

Remark 3 (the numbers c_k). For $k \leq 20$ the denominator of c_k divides the numerator of the Bernoulli number B_{6k} , with equality for $k \leq 4$ (the quotient first differs from 1 at $k = 5$, where it is 5). Neither the numerators nor the denominators of c_k occur in the OEIS (searched September 2026).

Remark 4 (what is classical). For integer s , (4) is the Lipschitz (Fourier) expansion of the lattice sum defining $E_s(\rho)$, and Corollary 1 is its sector form. Ramanujan’s (9.2.5), however, is stated for real s , where no modular interpretation of $L(s)$ is available; Theorem 1 supplies the missing K, λ there, and Theorem 2 holds for all complex s with $\text{Re } s > 2$ outside the degenerate classes. Lattice sums weighted by $\cos(m\theta)$ with an integer angular index m are known as angular lattice sums; see [6, Ch. 3] and the references there. In $R(s)$ the angular frequency equals the exponent s , is in general not an integer, and the sum runs over the sector $a, b \geq 1$ only; for $s \notin 6\mathbb{Z}$ it is therefore not one of these sums.

5 Numerical evidence

All computations are reproducible with the scripts archived at [8] (Python, NumPy, mpmath, SymPy). The script `verify/verify_all.py` repeats the checks below as 14 pass/fail tests (exit code 0 if and only if all pass, about two seconds); in `verify/mutation_test.py` each of 17 deliberate alterations of a stated formula or constant makes the corresponding test fail. They are empirical checks in double or multiple precision, not rigorous error bounds.

- *Lemma 1.* All 35088 primitive w with $|a|, |b| \leq 120$: no violation of “exactly three”; unit orbit: two in H , boundary $\{\pm\omega\}$.
- *Fixed s .* At $s = 4.7, 5.5, 6, 7.5, 8.7, 9.25, 13.1$ the two sides of Theorem 1 agree to relative $8 \cdot 10^{-16}$ – $3.5 \cdot 10^{-12}$ (3.8 million coprime pairs, left side to 30 digits); other natural choices of K, λ miss by $5 \cdot 10^{-6}$ – $1.5 \cdot 10^{-1}$ (at $s = 6$ the choice $K = \mu + 2\nu$, $\lambda = \mu$ also agrees: its angles are $\pi/3 - \psi_w$, and $\cos(6(\pi/3 - \psi)) = \cos(6\psi)$). The script `verify/numeric_check_K_lambda.py` also includes the degenerate values $s = 4, 10$, where every choice agrees.

- *Random s .* For 40 values $s \sim U(2.1, 20)$ the lattice sum was truncated at $|w| \leq 2400$ with the tail correction $T(X) = \frac{12}{\sqrt{3}\pi^2} \cdot \frac{2\sin(\pi s/6)}{s} \cdot \frac{X^{2-s}}{s-2}$; in all 40/40 cases the relative discrepancy was at most ten times the estimated error $|R_X - R_{X/2}|$ (plus 10^{-13}); the control $K = \mu - \nu$, $\lambda = \mu$ never came closer than $2.3 \cdot 10^{-9}$.
- *Complex s .* $s = 5 + 3i$: $2.0 \cdot 10^{-15}$; $s = 7.3 - 2i$: $5.4 \cdot 10^{-17}$; $s = 3.5 + i$: $5.8 \cdot 10^{-11}$.
- *Corollary 1.* For $k \leq 5$ the closed form agrees with Theorem 2 to $2 \cdot 10^{-40}$ (40 digits); see Table 1.

k	c_k	$R(6k)$
1	1	$0.039486299736351458 \dots$
2	250/691	$0.0013602406593698314 \dots$
3	5500/43867	$5.0757704420300926 \dots \times 10^{-5}$
4	10285000/236364091	$1.88165499540916937 \dots \times 10^{-6}$
5	26021050000/1723168255201	$6.96920718649738464 \dots \times 10^{-8}$

Table 1: Closed forms of Corollary 1, checked against Theorem 2.

Use of generative AI

Claude Opus 5.5 (Anthropic, model identifier `claude-opus-5-5`), accessed through Claude Code in September 2026, was used to locate the open question in [2], to propose the identification of K and λ , to carry out the symbolic and high-precision numerical computations reported here (with SymPy and mpmath) under the author’s direction, and to help prepare the text. It was used to verify every stated number independently at high precision. The author directed the work, checked all results and takes full responsibility for the content.

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