

THE TRINOMIALS $1 + ax^3 + x^k$ ARE NOT UNFAIR FACTORS OF 0–1 POLYNOMIALS

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ABSTRACT. The unfair 0–1 polynomial conjecture asserts that if a polynomial with coefficients in $\{0, 1\}$ factors as a product of two monic polynomials with nonnegative real coefficients, then both factors are 0–1 polynomials. We prove that for every $k \geq 4$ with $k \not\equiv 0 \pmod{3}$ and every $0 < a < 1$, the trinomial $1 + ax^3 + x^k$ does not divide any 0–1 polynomial with a nonzero cofactor having nonnegative coefficients; by reciprocity the same holds for $1 + ax^{k-3} + x^k$. The proof analyses the forced recurrence $q_n + aq_{n-3} + q_{n-k} = r_n \in \{0, 1\}$. An exact class formula for the coefficients along a distinguished “main branch” and elementary deviation lemmas reduce the problem to the main branch; for $a \in (1/2, 1)$ this is settled by hand, and for $a \in (0, 1/2]$ and $k \geq 67$ by a single rigorous interval-arithmetic certificate that is uniform in k . The remaining range $k \leq 149$ is covered by exhaustive computer certificates with exact algebraic treatment of isolated parameters.

1. INTRODUCTION

A polynomial is a 0–1 *polynomial* if all its coefficients lie in $\{0, 1\}$. The following question was raised on MathOverflow [7].

Conjecture 1.1 (Unfair 0–1 polynomial conjecture). *Let $P, Q \in \mathbb{R}[x]$ be monic with nonnegative coefficients. If PQ is a 0–1 polynomial, then P and Q are 0–1 polynomials.*

Following the literature, we call a polynomial P with nonnegative coefficients an *unfair factor* if it is not a 0–1 polynomial and PQ is a 0–1 polynomial for some nonzero Q with nonnegative coefficients. Ghidelli [3] showed that $1 + ax^2 + x^5$ ($0 < a < 1$) is not an unfair factor. Hare [4] gave an algorithm that rules out all candidate factors of degree at most 15 with the exception of 975 candidates (out of 7141686 cases); the list of exceptions does not appear to be publicly available. Hare also observed that a computation up to index 100 suffices to show that $1 + ax^3 + x^5$ is not an unfair factor; the case $(b, k) = (3, 5)$ of our theorem is therefore due to Hare. Hare’s Theorem 1.1 [4] is the reciprocity principle used below. Dvorsky [2] treats the family $1 + ax^2 + x^k$ for odd $k \geq 341$, the bounded odd range

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$7 \leq k \leq 339$ being taken from a companion manuscript [1]; that manuscript is not public and we have not seen it (its metadata are as cited in [2]). On the computational side, Zhang reports a verification of the conjecture for $\deg(PQ) \leq 66$ [11] (the README of the accompanying repository [12] states $\deg(PQ) \leq 45$). For related work on an ε -version of the problem see [6].

Our main result concerns the next family of trinomials.

Theorem 1.2. *Let $k \geq 4$ with $k \not\equiv 0 \pmod{3}$ and let $0 < a < 1$. Then $P(x) = 1 + ax^3 + x^k$ does not divide any 0–1 polynomial R with $R = PQ$, where $Q \neq 0$ has nonnegative real coefficients.*

Corollary 1.3. *Under the same hypotheses, $1 + ax^{k-3} + x^k$ is not an unfair factor.*

Proof. If $PQ = R$ then $P^*Q^* = R^*$, where $f^*(x) = x^{\deg f}f(1/x)$; Q^* has nonnegative coefficients and R^* is a 0–1 polynomial. Since $P^* = 1 + ax^{k-3} + x^k$, the claim follows from Theorem 1.2. This is Hare’s Theorem 1.1 [4]. $\square$

Remark 1.4. If $k \equiv 0 \pmod{3}$ the statement is trivial; see Remark 4.6. The families $(2, k) \cong (k-2, k)$ of [2] and $(3, k) \cong (k-3, k)$ meet only for $k = 5$, the case of [3]. Since Hare’s computation [4] covers all candidate factors of degree ≤ 15 apart from 975 unpublished exceptions, the cases $k \leq 15$ of Theorem 1.2 may be partly covered there; the theorem is certainly new for $k \geq 16$.

Strategy. We fix a and k and study all sequences (q_n) compatible with the recurrence $q_n = r_n - aq_{n-3} - q_{n-k}$, $r_n \in \{0, 1\}$, $q_n \geq 0$ (Section 2). They form a binary tree that branches only at indices where $aq_{n-3} + q_{n-k} = 0$. There is a distinguished *main branch* whose coefficients are given in closed form by an exact class formula, Lemma 3.2 (Section 3). Elementary lemmas show that every branch deviating from the main branch dies (Section 4), provided the main branch has a positivity property at its forced indices and eventually dies. This is checked by hand for $a \in (1/2, 1)$ (Section 5) and by one interval-arithmetic certificate valid for all $k \geq 67$ and $a \in (0, 1/2]$ (Section 6). The range $4 \leq k \leq 149$ is covered by direct computer certificates (Section 7).

Notation. For a power series $G(a) = \sum_{i \geq 0} g_i a^i$ and an integer L we write $T_L[G] = \sum_{i=0}^L g_i a^i$, with $T_L[G] = 0$ for $L < 0$. Note $(1+a)^{-(p+1)} = \sum_{i \geq 0} (-1)^i \binom{p+i}{p} a^i$. Throughout, $c \in \{1, 2\}$ denotes $k \bmod 3$ and residue classes are taken modulo 3: $C_0 = \{n \equiv 0\}$, $C_k = \{n \equiv k\}$, $Z = \{n \equiv -k\}$; these are the three distinct classes.

2. THE FORCED RECURRENCE

Definition 2.1. Fix $a \in (0, 1)$ and k . A *path* is a sequence $r = (r_n)_{n \geq 0} \in \{0, 1\}^{\mathbb{N}}$ with $r_0 = 1$. It determines $q_n = q_n(r)$ by $q_n = 0$ for $n < 0$ and

$$(1) \quad q_n + aq_{n-3} + q_{n-k} = r_n \quad (n \geq 0),$$

equivalently $(1 + ax^3 + x^k) \sum q_n x^n = \sum r_n x^n$ as formal power series. Put $S_n = aq_{n-3} + q_{n-k}$. The path *dies at N* if $q_n \geq 0$ for $n < N$ and $q_N < 0$.

Lemma 2.2. *If $1 + ax^3 + x^k$ divides a 0–1 polynomial with a nonzero cofactor with nonnegative coefficients, then some path never dies.*

Proof. Let $PQ = R$. Dividing Q and R by the largest power of x dividing Q we may assume $q_0 > 0$. Then $r_0 = q_0 > 0$, so $r_0 = q_0 = 1$. The coefficient sequences of Q and R (extended by zeros) satisfy (1) with $q_n \geq 0$ for all n . $\square$

Given $q_0, \dots, q_{n-1} \geq 0$, the choice $r_n = 0$ keeps $q_n \geq 0$ iff $S_n = 0$, and $r_n = 1$ keeps $q_n \geq 0$ iff $S_n \leq 1$. In particular the paths that are alive up to index n branch only at indices with $S_n = 0$; if $S_n > 0$ then $r_n = 1$ is forced. By Lemma 2.2, Theorem 1.2 follows once we show that *every path dies*.

Lemma 2.3 (Linear response). *Let $g_\Delta = [x^\Delta](1 + ax^3 + x^k)^{-1}$. For two paths r, r' , $q_t(r') - q_t(r) = \sum_{s \leq t} (r'_s - r_s) g_{t-s}$. For $0 \leq \Delta < k$ we have $g_\Delta = (-a)^{\Delta/3}$ if $3 \mid \Delta$ and $g_\Delta = 0$ otherwise.*

Proof. The first claim is linearity of $Q = R/P$. Writing

$$(2) \quad \frac{1}{1 + ax^3 + x^k} = \sum_{p \geq 0} (-1)^p \frac{x^{pk}}{(1 + ax^3)^{p+1}},$$

only $p = 0$ contributes to degrees $< k$, and $(1 + ax^3)^{-1} = \sum_i (-a)^i x^{3i}$. $\square$

3. THE MAIN BRANCH AND THE CLASS FORMULA

From now on $k \geq 4$ and $k \not\equiv 0 \pmod{3}$. Let $\mathcal{F} = Z \cap (k, 2k]$ (the *free set*); its least element is $d_0 = k + 1$ if $c = 1$ and $d_0 = k + 2$ if $c = 2$, and its largest element is $2k$.

Definition 3.1. The *main branch* is the path r^* with $r_n^* = [3 \mid n]$ for $n < k$, $r_k^* = 1$, $r_n^* = 0$ for $n \in \mathcal{F}$, $r_n^* = 1$ for $n \in (k, 2k] \setminus \mathcal{F}$, and $r_n^* = 1$ for $n > 2k$. We write $q_n^* = q_n(r^*)$ and S_n^* accordingly.

On each residue class the support of r^* is a tail: $\{n \in C_0 : n \geq 0\}$, $\{n \in C_k : n \geq k\}$ and $\{n \in Z : n \geq 2k + 3\}$. Define the thresholds $\tau = 0, k, 2k + 3$ for M in C_0, C_k, Z respectively.

Lemma 3.2 (Class formula). *For every $N \geq 0$, with $M_p = N - pk$ and $L_p = (M_p - \tau(M_p))/3$,*

$$q_N^* = \sum_{p \geq 0, M_p \geq 0} (-1)^p T_{L_p}[(1 + a)^{-(p+1)}],$$

where terms with $L_p < 0$ vanish. The right-hand side is a polynomial in a .

Proof. By (2), $q_N^* = \sum_p (-1)^p \sum_{i \geq 0} \binom{p+i}{p} (-a)^i r_{M_p-3i}^*$. All indices $M_p - 3i$ lie in the class of M_p , and $r_{M_p-3i}^* = 1$ iff $M_p - 3i \geq \tau(M_p)$, i.e. iff $i \leq L_p$ (note $M_p \equiv \tau(M_p)$, so L_p is an integer). $\square$

The formula was checked against the recurrence with exact rational polynomial arithmetic for $4 \leq k \leq 120$ and all $N \leq 5k + 12$ (24999 values, no discrepancy); see Section 9.

Corollary 3.3. *For $t \in C_0$ with $0 \leq t < 3k$, $q_t^* = T_{t/3}[(1+a)^{-1}] = \frac{1 - (-a)^{t/3+1}}{1+a} > 0$ for all $a \in (0, 1)$.*

Proof. For $p = 1$, $M_1 = t - k \in Z$ and $M_1 \geq 2k + 3$ would need $t \geq 3k + 3$; for $p = 2$, $M_2 = t - 2k \in C_k$ and $M_2 \geq k$ would need $t \geq 3k$; for $p \geq 3$, $M_p < 0$. $\square$

Lemma 3.4. *For all $a \in (0, 1)$: $q_n^* = 0$ for $n < k$ with $n \notin C_0$; $q_k^* = 0$; and for $n \in \mathcal{F}$ one has $q_{n-3}^* = q_{n-k}^* = 0$, hence $S_n^* = 0$ and $q_n^* = 0$.*

Proof. For $n < k$, (1) reads $q_n = r_n - aq_{n-3}$, so classes C_k and Z stay zero below k under r^* . Then $q_k^* = 1 - aq_{k-3}^* - q_0^* = 0$ since $k - 3 \in C_k$. For $n \in \mathcal{F}$, $n - 3 \in Z$ is either $< k$ or in $\mathcal{F}$ (induction), and $n - k \in C_k \cap (0, k]$, where q^* vanishes. $\square$

4. DEVIATIONS FROM THE MAIN BRANCH

Lemma 4.1 (Window $n \leq k$). *Let $a \in (0, 1)$ and let r be a path alive up to index k . Then $r_{3j} = 1$ and $q_{3j} = \sum_{i=0}^j (-a)^i \in (0, 1]$ for $3j < k$; $q_n = 0$ for $n < k$, $n \in C_k$; and $r_k = 1$, $q_k = 0$.*

Proof. For $n < k$ we have $q_n = r_n - aq_{n-3}$. If $q_{n-3} \in (0, 1]$ then $r_n = 0$ would give $q_n < 0$, so $r_n = 1$ and $q_n = 1 - aq_{n-3} \in [1 - a, 1) \subset (0, 1]$. Starting from $q_0 = 1$ this gives the claim on C_0 . If some $n < k$ in C_k had $q_n > 0$, the same propagation would give $q_{k-3} > 0$ (as $k - 3 \in C_k$), and then $q_k = r_k - aq_{k-3} - 1 < 0$. Hence $q_{k-3} = 0$ and $q_k = r_k - 1 \geq 0$ forces $r_k = 1$, $q_k = 0$. $\square$

So a path alive at index k agrees with r^* below k except possibly on $Z \cap [1, k)$, where r^* vanishes.

Lemma 4.2 (C_0 -shadowing). *Let $a \in (0, 1)$, let r agree with r^* on $[0, d]$ for some $d \geq 1$, and let $T \leq \min(d + k - 1, 3k - 1)$. If r is alive up to T , then for every $t \in C_0 \cap [0, T]$ we have $r_t = 1$ and $q_t(r) = q_t^* > 0$.*

Proof. Induction on $t \in C_0$. For $t < d$ there is nothing to prove. Let $d \leq t \leq T$. By induction (or Lemma 4.1 if $t - 3 < d$) $q_{t-3}(r) = q_{t-3}^* > 0$ (Corollary 3.3, as $t - 3 < 3k$), so $r_t = 1$. By Lemma 2.3, $q_t(r) - q_t^* = \sum_{s=d}^t (r_s - r_s^*)g_{t-s}$ with $0 \leq t - s \leq T - d < k$; thus $g_{t-s} \neq 0$ only for $s \in C_0$, where $r_s = 1 = r_s^*$ by induction. Positivity is Corollary 3.3. $\square$

Proposition 4.3 (Late starts). *Let $a \in (0, 1)$. If a path r alive at k has $r_{n_0} = 1$ for some $n_0 \in Z \cap [1, k)$, then it dies at an index $\leq k + n_0$.*

Proof. Take n_0 minimal. Then r agrees with r^* on $[0, n_0)$ (Lemma 4.1) and $q_{n_0}(r) = 1 - aq_{n_0-3}(r) = 1$, since $n_0 - 3 \in Z$ lies below n_0 . Now $k + n_0 \equiv k - k \equiv 0$, so $t = k + n_0 - 3 \in C_0$, and $T = k + n_0 - 3 \leq \min(n_0 + k - 1, 3k - 1)$. If r is alive up to T , Lemma 4.2 gives $q_t(r) = q_t^* > 0$, and then $q_{k+n_0}(r) = r_{k+n_0} - aq_t(r) - 1 < 0$. $\square$

Proposition 4.4 (Free deviations). *Let $a \in (0, 1)$ and let r agree with r^* on $[0, d_1)$ with $d_1 \in \mathcal{F}$ and $r_{d_1} = 1$. Then r dies at an index $\leq d_1 + k$.*

Proof. By Lemma 3.4, $q_{d_1}(r) = 1 - aq_{d_1-3}^* - q_{d_1-k}^* = 1$. The index $t = d_1 + k - 3$ lies in C_0 and $t \leq 3k - 3$, so with $T = t \leq \min(d_1 + k - 1, 3k - 1)$ Lemma 4.2 gives $q_t(r) = q_t^* > 0$ if r is alive up to t . Hence $q_{d_1+k}(r) = r_{d_1+k} - aq_t(r) - 1 < 0$. $\square$

Corollary 4.5 (Reduction to the main branch). *Let $a \in (0, 1)$. Suppose the main branch dies at some $N > k$ and $S_n^* > 0$ for every $n \in (k, N) \setminus \mathcal{F}$. Then every path dies.*

Proof. Let r be a path alive at k (otherwise we are done); by Lemma 4.1, r agrees with r^* on $[0, k]$ outside $Z \cap [1, k)$. If r differs from r^* on $Z \cap [1, k)$, Proposition 4.3 applies. Otherwise let $n > k$ be the first index with $r_n \neq r_n^*$, if any. If $n \in \mathcal{F}$ then $r_n = 1$ and Proposition 4.4 applies. If $n \notin \mathcal{F}$ and $n < N$, then $r_n = 0$ while $S_n(r) = S_n^* > 0$, so $q_n(r) < 0$. If no such $n < N$ exists, r agrees with r^* below N ; since $q_N^* = 1 - S_N^* < 0$ forces $S_N^* > 1$, both choices of r_N give $q_N < 0$. $\square$

Remark 4.6. If $k \equiv 0 \pmod{3}$, the argument of Lemma 4.1 gives $q_{k-3} > 0$ in every path alive at $k - 1$ (now $k - 3 \in C_0$), so every path dies at $n = k$.

5. THE RANGE $1/2 < a < 1$

Lemma 5.1 (Lemma A). *Let $k \geq 7$, $k \not\equiv 0 \pmod{3}$ and $a \in (1/2, 1)$. Then $q_{k+3}^* = a$ and $q_{k+6}^* = a(1 - 2a) < 0$, and every path dies.*

Proof. By Lemmas 4.1 and 3.4, $q_k^* = 0$, $q_3^* = 1 - a$, and $q_6^* = 1 - a + a^2$ since $6 < k$. Then $q_{k+3}^* = 1 - a \cdot 0 - (1 - a) = a$ and $q_{k+6}^* = 1 - a^2 - (1 - a + a^2) = a(1 - 2a) < 0$. The indices $n \in (k, k + 6) \setminus \mathcal{F}$ are $k + 3 \in C_k$, with $S_{k+3}^* \geq q_3^* > 0$, and the C_0 -indices among $k + 1, \dots, k + 5$, with $S_n^* \geq aq_{n-3}^* > 0$ by Corollary 3.3. Hence the main branch dies at some $N \leq k + 6$ and $S_n^* > 0$ on $(k, N) \setminus \mathcal{F}$; Corollary 4.5 concludes. $\square$

Remark 5.2. One can also see directly that every path whose first deviation from r^* is at an index $s > k$ dies at $k + 6$: the free indices below $k + 6$ are d_0 and $d_0 + 3$, so the relevant differences $k + 3 - s$ and $k + 6 - s$ lie in $\{1, 2, 4, 5\}$ and the corresponding g_Δ vanish by Lemma 2.3. Thus $q_{k+6} \leq a(1 - 2a) < 0$ whatever r_{k+6} is.

Remark 5.3 (Formal verification). Three of the hand arguments have been checked in Lean 4 (v4.34.1, with Mathlib), for an arbitrary path $r \in \{0, 1\}^{\mathbb{N}}$ and without coprimality hypotheses; the theorems are stated for general

b and are used here with $b = 3$ (file `lean/Trinom/Trinom/Ogish.lean`): `A_b_star` gives the conclusion “every path dies” of Lemma 5.1 (no path has $q_n \geq 0$ for all $n \leq k + 6$); `E1` gives the step of Lemma 4.1 that a positive coefficient in C_k below k forces $q_k < 0$; and `dies_of_dvd` gives Remark 4.6. The file contains no `sorry`, and `#print axioms` reports only `propext`, `Classical.choice` and `Quot.sound`. The remaining lemmas and the computer certificates are not formalised.

6. THE RANGE $0 < a \leq 1/2$, $k \geq 67$: A UNIFORM CERTIFICATE

Proposition 6.1 (Lemma U). *Let $k \geq 67$, $k \not\equiv 0 \pmod{3}$ and $a \in (0, 1/2]$. Then the main branch dies at some $N \leq 5k + 12$, and $S_n^* > 0$ for every $n \in [k, N) \setminus \mathcal{F}$.*

The proof is a rigorous computation that is independent of k ; we first explain why finitely many computations suffice.

6.1. Reduction to finitely many types. Write $N = jk + m$ with $0 \leq m < k$. Then $M_p = (j - p)k + m$ lies in the class $(j - p)c + m \pmod{3}$, and Lemma 3.2 gives $L_p = (\alpha_p k + m - \beta_p)/3$, where $(\alpha_p, \beta_p) = (j - p, 0)$, $(j - p - 1, 0)$ or $(j - p - 2, 3)$ for M_p in C_0 , C_k , Z respectively. Terms with $\alpha_p < 0$ vanish. Terms with $\alpha_p = 0$ (“live” terms) have $L_p = (m - \beta_p)/3$, independent of k . Terms with $\alpha_p \geq 1$ (“small” terms) have $L_p \geq (k - 3)/3$.

Lemma 6.2. *Let $p \geq 0$, $L \geq p$ and $a \in (0, 1/2]$. Then $|(1 + a)^{-(p+1)} - T_L[(1 + a)^{-(p+1)}]| \leq \binom{p+L+1}{p} a^{L+1}$, and the right-hand side is non-increasing in L .*

Proof. The series $\sum_i (-1)^i \binom{p+i}{p} a^i$ alternates, and the ratio of consecutive terms is $\frac{p+i+1}{i+1} a \leq \frac{p+i+1}{2(i+1)} \leq 1$ for $i \geq p - 1$; apply the Leibniz bound. The same ratio computation with $i = L + 1$ gives monotonicity. $\square$

Consequently, for $j \leq 5$ and $k \geq K$,

$$q_{jk+m}^*(a) = \frac{\mathcal{N}_{j,m}^{(c)}(a)}{(1+a)^6} + E,$$

$$|E| \leq \sum_{p: \alpha_p \geq 1} \binom{p + \Lambda + 1}{p} a^{\Lambda+1}, \quad \Lambda = \lfloor (K - 3)/3 \rfloor,$$

where $\mathcal{N}_{j,m}^{(c)}$ is an explicit polynomial: each small term contributes $(-1)^p (1 + a)^{5-p}$ and each live term $(-1)^p T_{L_p}[(1 + a)^{-(p+1)}] (1 + a)^6$. It depends on k only through $c = k \pmod{3}$. For $m > M_0$ (we use $M_0 = 60$) the live terms are also replaced by $(1 + a)^{-(p+1)}$, with error at most $\binom{p+\Lambda_0+1}{p} a^{\Lambda_0+1}$, $\Lambda_0 = 19$; the resulting type depends on m only through $m \pmod{3}$. Hence, for fixed c , the main branch on the windows $j = 1, \dots, 5$ is described by finitely many *types*, each a rational function plus an explicit error bound.

6.2. The certificate. The script `retsenziya/uniform_sertifikat.py` does the following for each $c \in \{1, 2\}$ and each of the 64 intervals $I = [i/128, (i+1)/128]$, $0 \leq i \leq 63$. It walks the main branch through the positions (j, m) in increasing order: $j = 1, \dots, 4$ with $m = 0, \dots, M_0 + 3$ individually followed by the three residues of $m > M_0 + 3$ (“generic” positions), and $j = 5$ with $m = 0, \dots, 12$; this covers every $n \in [k, 5k + 12]$ when $k > M_0 + 3$. For each type it decides the sign on I rigorously: after removing the largest power a^v dividing $\mathcal{N}$, it encloses $\mathcal{N}(a)/a^v$ on I by Arb ball arithmetic (`arb_poly` evaluated on the ball $[lo, hi]$) and compares with the enclosure $(1 + hi)^6 \sum C hi^{\lambda-v}$ of the error term; a sign is accepted only if it is strict on the whole closed interval, otherwise the type is treated as ambiguous (possibly zero). Then:

(S1) at every non-free position before the death, q_{n-3}^* or q_{n-k}^* must be certified strictly positive on I ;

(S2) some position must be certified strictly negative on I (the death).

Failing intervals would be bisected (up to depth 40); none had to be. Since the walk is in increasing order, (S1) is checked at every non-free n up to the first certified negative value, which bounds the actual death index from above. Because intervals are closed and signs are strict on them, isolated parameters (e.g. roots of some q_n^*) need no separate treatment.

The certificate was run with $K = 151$ and with $K = 67$ (the condition $k > M_0 + 3 = 63$ holds). In both runs and in both classes all 64 intervals were certified without subdivision, the deaths being of the four types

$$3k + 12 \text{ (2 intervals), } \quad 4k + 15 \text{ (1), } \quad 5k + 3 \text{ (59), } \quad 5k + 12 \text{ (2).}$$

A run takes about 0.7 seconds. This proves Proposition 6.1, the interval $[63/128, 1/2]$ covering $a = 1/2$.

Remark 6.3 (Sanity checks). Three deliberate mutations of the certificate fail as they should: checking (S1) also on the free set makes all 128 intervals fail; flipping the sign of the live terms changes the outcome; inflating the error bound by 10^{12} makes 44 intervals fail. A comparison of the type signs with exactly computed q_N^* at 1200 random parameters for $k = 151, 152, 301$ found no contradiction; this comparison uses the same type model and is therefore not an independent check.

Remark 6.4 (Limit shapes; not used in the proof). Lemma 3.2 explains the four death types. For instance, for $k \geq 13$,

$$q_{5k+3}^* = T_{k+1}[G_1] + (-1)^k a^{k+1}, \quad q_{5k+12}^* = T_{k+4}[G_2] + (-1)^{k+1} a^{k+4},$$

with $G_1 = a - 1 + (1+a)^{-1} - (1+a)^{-2} + (1+a)^{-3} = a(a^3 + 2a^2 + a - 1)/(1+a)^3$ and $G_2 = -56a^4 + 41a^3 - 16a^2 + 5a - 1 + (1+a)^{-1} - (1+a)^{-2} + (1+a)^{-3}$ (verified by computer for $13 \leq k \leq 300$). Numerically (coefficients compared up to $k = 40$), the k -independent limits of q_n^* at $n = 5k + 3, 5k + 12, 4k + 15, 3k + 12$ are negative for $a < \alpha$, $a > \beta$, $a > 0.47098\dots$ and $a > 0.47681\dots$ respectively, where $\alpha = 0.4655712318\dots$ and $\beta = 0.4484743767\dots$ are the

roots in $(0, 1)$ of $1 - a - 2a^2 - a^3$ and of the numerator of G_2 . Since $(0, \alpha) \cup (\beta, 1/2] = (0, 1/2]$, the main branch always dies; the certificate makes this rigorous uniformly in k .

7. SMALL k AND ISOLATED PARAMETERS

Proposition 7.1. *For $4 \leq k \leq 149$, $k \not\equiv 0 \pmod{3}$, and every $a \in (0, 1)$, every path dies.*

This is established by an exhaustive computer certificate over the whole parameter interval $(0, 1)$ (script `simvolik_isbot_flint.py` with `GCD_TEKSHIR=1`). The program performs a depth-first search over paths in which each node carries a subinterval $I \subset (0, 1)$ and the exact polynomials $q_0, \dots, q_{n-1} \in \mathbb{Q}[a]$ valid on I (FLINT `fmpq_poly`). At index n :

- the branch $r_n = 0$ is opened when q_{n-3} and q_{n-k} are both the zero polynomial;
- for $r_n = 1$, I is split according to the sign of $q_n = 1 - aq_{n-3} - q_{n-k}$. Real roots are counted exactly with Sturm sequences (PARI `polsturm`); approximate roots from `polrootsreal` are enclosed in rational boxes of width about 2^{-40} whose root counts are confirmed by `polsturm`, with exact bisection as a fallback. An Arb ball evaluation that excludes 0 on I certifies a constant sign without Sturm. Subintervals where $q_n < 0$ die; the others, including the narrow boxes containing roots, are continued (conservatively);
- parameters where $r_n = 0$ is possible without q_{n-3}, q_{n-k} vanishing identically are isolated algebraic numbers: common roots of q_{n-3} and q_{n-k} (gcd, factored by PARI `factor`) and roots of q_n falling exactly on a subinterval endpoint are collected.

At each collected algebraic parameter α the complete tree, including all $r_n = 0$ branches, is rerun in exact arithmetic in $\mathbb{Q}(\alpha)$ (`nuqtaviy.py`); signs are decided by refining an isolating interval of the minimal polynomial with Sturm sequences. A run is accepted if no branch reaches the depth bound N , no terminating cofactor is detected, and all isolated parameters die.

For $k \leq 60$ the runs (depth bound $N = 2000$) are recorded in `retsenziya/toliq_census.log`; all 38 runs are accepted, 36 isolated parameters were collected in total and all die, and for $k \geq 13$ the maximal depth reached is exactly $5k + 12$. For $61 \leq k \leq 149$ (60 values, $N = 4000$; `retsenziya/b3_61_150.log`) all 60 runs are accepted, the maximal depth is exactly $5k + 12$ in each run, 60 isolated parameters were collected in total (for instance $a = 1/2$) and all of them die; each run takes at most about 7 seconds. The range $67 \leq k \leq 149$ is thus covered twice (here and by Proposition 6.1).

Exact check at root points for $151 \leq k \leq 250$. As an independent confirmation of the treatment of isolated parameters in Proposition 6.1, for each $k \in [151, 250]$, $k \not\equiv 0 \pmod{3}$ (67 values), the script `retsenziya/ildiz_nuqta.py`

computes all main-branch polynomials q_n^* , $n \leq 5k + 12$, collects all their roots in $(0, 1/2]$, and runs the complete exact tree in $\mathbb{Q}(\alpha)$ at each root (depth bound $5k + 60$). In every case there are exactly six such roots, the same for all k :

$$\begin{aligned} &0.4484743767\dots, \quad 0.4655712319\dots, \quad 0.4709761380\dots, \\ &0.4768143659\dots, \quad 0.4874322956\dots, \quad 1/2 \end{aligned}$$

(the last being the root of $q_{k+6}^* = a(1 - 2a)$). All 402 point trees die, none survives, and the maximal depth is exactly $5k + 12$ for every k ; in total 1895851 branchings were explored (about 19971 s of running time summed over the runs). With the depth bound lowered to $5k + 10$, two points are correctly reported alive. An independent re-implementation produced the same outcome for $k = 151, 152, 154, 155, 301, 302$.

8. PROOF OF THEOREM 1.2

Let $k \geq 4$, $k \not\equiv 0 \pmod{3}$, $a \in (0, 1)$. By Lemma 2.2 it suffices to show that every path dies. If $k \leq 149$ this is Proposition 7.1. If $k \geq 67$ and $a \in (1/2, 1)$, it is Lemma 5.1. If $k \geq 67$ and $a \in (0, 1/2]$, Proposition 6.1 gives the hypotheses of Corollary 4.5 (note $n = k \notin \mathcal{F}$ and $S_k^* = aq_{k-3}^* + q_0^* = 1 > 0$), which concludes. $\square$

The proof depends on the correctness of python-flint [10] (FLINT [8] and its ball-arithmetic component Arb [5]: `fmq_poly`, `arb`, `arb_poly`) and of PARI/GP [9] (`polsturm`, `polrootsreal`, `factor`), both standard libraries.

9. COMPUTER VERIFICATION AND REPRODUCIBILITY

All scripts are in the project directory (paths relative to it) and were run with a Python environment providing python-flint 0.9.0 and cypari2 2.2.4 (PARI 2.17.2), denoted `python` below.

- (1) *Class formula* (Lemma 3.2): `python retsenziya/oina_formula.py 4 120` compares the formula with the recurrence for all $N \leq 5k + 12$; output: 24999 values, 0 discrepancies.
- (2) *Uniform certificate* (Proposition 6.1): `python retsenziya/uniform_sertifikat.py 151 60` and `python retsenziya/uniform_sertifikat.py 67 60`; output per class: 64 intervals checked, 0 failed, death types $\{(3, 12):2, (4, 15):1, (5, 3):59, (5, 12):2\}$, where (j, m) stands for $n = jk + m$.
- (3) *Small k* (Proposition 7.1): `GCD_TEKSHIR=1 python simvolik_isbot_flint.py 3 k 2000` for $k \leq 60$ (driver `retsenziya/toliq_census.sh`, log `retsenziya/toliq_census.log`) and `GCD_TEKSHIR=1 python simvolik_isbot_flint.py 3 k 4000` for $61 \leq k \leq 149$ (driver `retsenziya/b3_61_150.sh`, log `retsenziya/b3_61_150.log`).

Each run prints a JSON record; acceptance is the field `isbot_toliq`.

- (4) *Root points*: `python retsenziya/ildiz_nuqta.py k` for $151 \leq k \leq 250$ (driver `retsenziya/ildiz_nuqta.sh`, log `retsenziya/ildiz_nuqta.log`).
- (5) *Formal verification* (Remark 5.3): `lake build` in `lean/Trinom` (toolchain `leanprover/lean4:v4.34.1`, Mathlib v4.34.1); the build completes without errors and the files `Trinom/Basic.lean` and `Trinom/Ogish.lean` contain no `sorry`.

Availability. The scripts listed above (`simvolik_isbot_flint.py`, `nuqtaviy.py` and the files under `retsenziya/`), the complete logs of all runs reported in Sections 6 and 7, and the Lean sources are available together with this paper. Each certificate run recomputes its verdict and reads no stored data (the scripts only write their output), so the logs serve only for comparison. Timings quoted in the text are wall-clock times of single runs and are given for orientation only. All of this material is deposited at Zenodo, doi: [10.5281/zenodo.23311094](https://doi.org/10.5281/zenodo.23311094), and mirrored at <https://github.com/selloriybrendi/zero-one-trinomial-b3>. All runs were made on an Apple M5 (10 cores, 16 GB RAM) under macOS, with python-flint 0.9.0 and PARI/GP 2.17.2.

USE OF AI TOOLS

The AI assistant Claude (Anthropic) was used in the preparation of this work: to write and run the computational scripts, including the certificate programs of Sections 6 and 7, to draft the Lean formalisation of Remark 5.3, to search and check the literature, and to draft the text of this paper. The mathematical claims do not rest on the output of the AI tool: the computer-assisted parts are checked by the exact and interval arithmetic of FLINT/Arb [8, 5] and PARI/GP [9], and the formalised parts by the Lean proof checker. The author has checked all arguments, computations and references in this paper and takes full responsibility for its content.

10. OUTLOOK

For general odd b the class formula extends: if $\gcd(b, k) = 1$ and the main branch is defined by $r_n = 1$ unless q_{n-b} and q_{n-k} both vanish identically, the thresholds are $\tau_0 = 0$, $\tau_1 = k$, $\tau_j = jk + b$ ($j \geq 2$) on the class $n \equiv jk \pmod{b}$, and $q_N = \sum_{p: L_p \geq 0} (-1)^p T_{L_p}[(1+a)^{-(p+1)}]$ with $L_p = (M_p - \tau_{j(M_p)})/b$. This was checked by computer (no discrepancy) for $b = 3$ ($4 \leq k \leq 60$), $b = 5$ ($6 \leq k \leq 40$) and $b = 7$ ($8 \leq k \leq 36$), but we do not state it here as a proved lemma for $b \geq 5$. The following is known only partially and is *not* proved here.

- *Finite ranges.* Exhaustive certificates of the type of Section 7 cover $b = 5$, $6 \leq k \leq 39$, and $b = 7$, $8 \leq k \leq 30$ ($k \not\equiv 0 \pmod{b}$).

- *Main branch, uniformly in k .* The analogue of the certificate of Section 6 shows that the main branch dies by index $7k + 5$ for $b = 5$, all $k \geq 201$, and by $9k + 7$ for $b = 7$, all $k \geq 1001$; the dominant death types are $(b + 2)k + b$ for $b = 3, 5, 7$. For $a \in (1/2, 1)$ and $k > 2b$ the main branch has $q_{k+b} = a$ and $q_{k+2b} = a(1 - 2a) < 0$.
- *Deviations.* The free-deviation lemma (Proposition 4.4) fails for $b \geq 5$: for $b = 5$, $k = 31$, a deviation at $n = 2k$ survives up to $13k + 30$ at an isolated $a \approx 0.2469$. A uniform treatment of deviating branches is missing.
- *Stability of deviation patterns.* For $b = 5$, describing the leaves of the full tree in window coordinates (offsets ≤ 30 from window ends kept exactly, middle offsets only modulo b), the sets of leaf patterns for $k = 66$ and $k = 71$ contain 53151 and 54617 patterns, with 53147 in common; only 4 patterns occur for $k = 66$ alone. The trees have 66706 and 85329 leaves, no live branch, and at most four deviations per path.

Conjecture 10.1. *For $b = 5$ (and plausibly every odd b), the deviation tree stabilises in window coordinates for large k , so that a finite catalogue of patterns, each with a uniform interval certificate, would prove the analogue of Theorem 1.2.*

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