

THE TWO LOST NOTES ON PAGE 54 OF RAMANUJAN'S LOST NOTEBOOK

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ABSTRACT. On page 54 of his lost notebook Ramanujan states two families of cubic theta-function identities and, in each family, writes “see note” next to one further claim ($A^3 - B^3$ in the first family, $A^3 + B^3$ in the second). Both notes are lost: Andrews and Berndt record that they have not been preserved. We show that both missing identities are special cases of another claim written on the *same page*, Ramanujan’s cubic circular summation (Entry 8.2.3 in Andrews–Berndt), and that in terms of the Borweins’ cubic theta function $a(q) = \sum_{m,n} q^{m^2+mn+n^2}$ they take the compact form

$$A^3 - B^3 = q^2 f^3(-q^3, -q^{12}) + f(-q^2, -q^3) a(q^5), \quad q(A^3 + B^3) = f^3(-q^6, -q^9) - f(-q, -q^4) a(q^5).$$

We also record an elementary second form obtained from Ramanujan’s other identities on the page, and computational evidence that, unlike the four recorded cube identities, the two missing combinations have no one- or two-term theta-*product* representation in a large explicit search space; the appearance of the binary theta series $a(q^5)$, which is not such a product, explains why these two claims were deferred to a note. All identities are certified to 2000 coefficients by a single integer-arithmetic script and independently at 50-digit precision. (Version 2; version 1 contained only the elementary form.)

1. INTRODUCTION

As usual, let

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1,$$

and $f(-q) := f(-q, -q^2) = (q; q)_{\infty}$. Page 54 of Ramanujan’s lost notebook [1] contains two groups of cubic identities for theta functions with quotient of arguments q^{15} ; they are treated as Entries 8.2.1 and 8.2.2 in Andrews and Berndt’s second volume [2, pp. 176–178].

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Entry 8.2.1. Let $A = f(-q^7, -q^8)$ and $B = qf(-q^2, -q^{13})$. Then

$$(8.2.6) \quad A + B = \frac{f(-q^2, -q^3)}{f(-q, -q^4)} f(-q^5),$$

$$(8.2.7) \quad A - B = f(-q^{2/3}, -q) + q^{2/3} f(-q^3, -q^{12}),$$

$$(8.2.8) \quad A^3 + B^3 = \frac{f(-q^6, -q^9)}{f(-q^3, -q^{12})} f^3(-q^5),$$

$$(8.2.9) \quad (A - B)^3 = \frac{f(-q^2, -q^3) f^3(-q, -q^4)}{f(-q^3, -q^{12})} + q^2 f^3(-q^3, -q^{12}).$$

After stating these, Ramanujan writes “ $A^3 - B^3$ *see note*.” Andrews and Berndt comment: “Either Ramanujan never wrote this note, or if he did, it has not been preserved” [2, p. 177].

Entry 8.2.2. Let $A = f(-q^4, -q^{11})$ and $B = qf(-q, -q^{14})$. Then

$$(8.2.10) \quad A - B = \frac{f(-q, -q^4)}{f(-q^2, -q^3)} f(-q^5),$$

$$(8.2.11) \quad A + B = -\frac{1}{q^{1/3}} \left\{ f(-q^{1/3}, -q^{4/3}) - f(-q^6, -q^9) \right\},$$

$$(8.2.12) \quad A^3 - B^3 = \frac{f(-q^3, -q^{12})}{f(-q^6, -q^9)} f^3(-q^5),$$

$$(8.2.13) \quad (A + B)^3 = -\frac{1}{q} \left\{ \frac{f(-q, -q^4) f^3(-q^2, -q^3)}{f(-q^6, -q^9)} - f^3(-q^6, -q^9) \right\}.$$

Here Ramanujan writes “ $A^3 + B^3$ *see note*,” and again “he apparently did not write this note, or, if he did, it has been lost” [2, p. 178].

To our knowledge the two missing notes have not been reconstructed in the literature. The cubic identities of page 54 were first proved by Son [11], and Andrews and Berndt state that in this section they “follow precisely Son’s beautiful proofs” [2, p. 173]; both “see note” remarks, with the statements that the notes are lost, appear inside that very section, and the only further content of [11] mentioned there is “some new modular equations” [2, p. 180]. We were unable to find any later source addressing the notes.

The main observation of this paper (Section 2) is that the two missing identities are hidden two lines further down the same page: they are special cases of Ramanujan’s cubic circular summation, Entry 8.2.3 of [2], whose right-hand side is the Borweins’ cubic theta function $a(q)$. Section 3 records a second, elementary form obtained from (8.2.7), (8.2.9), (8.2.11), (8.2.13). Section 4 shows that no one- or two-term theta-product form exists within a large explicit search space, and Section 6 describes the verification scripts.

2. THE NOTES AS CIRCULAR SUMMATIONS

On page 54 Ramanujan also states the following, proved by Son [4] and given as Entry 8.2.3 in [2, p. 178]: for $|ab| < 1$ and $q = ab$,

$$(1) \quad f^3(ab^2, a^2b) + af^3(b, a^3b^2) + bf^3(a, a^2b^3) = f(a, b) \left\{ \frac{f^9(-q)}{f^3(-q^3)} + 27q \frac{f^9(-q^3)}{f^3(-q)} \right\}^{1/3}.$$

With the Borweins' cubic theta functions [7, 8]

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \quad b(q) = \frac{f^3(-q)}{f(-q^3)}, \quad c(q) = 3q^{1/3} \frac{f^3(-q^3)}{f(-q)}, \quad a^3 = b^3 + c^3,$$

the brace in (1) equals $(b^3(q) + c^3(q))^{1/3} = a(q)$.

Theorem 1. *Let $A = f(-q^7, -q^8)$ and $B = qf(-q^2, -q^{13})$ (Entry 8.2.1). Then*

$$(2) \quad A^3 - B^3 = q^2 f^3(-q^3, -q^{12}) + f(-q^2, -q^3) a(q^5).$$

Proof. Take $a = -q^3$, $b = -q^2$ in (1), so $ab = q^5$. Then $f(ab^2, a^2b) = f(-q^7, -q^8) = A$, $af^3(b, a^3b^2) = -q^3 f^3(-q^2, -q^{13}) = -B^3$, $bf^3(a, a^2b^3) = -q^2 f^3(-q^3, -q^{12})$ and $f(a, b) = f(-q^2, -q^3)$. $\square$

Theorem 2. *Let $A = f(-q^4, -q^{11})$ and $B = qf(-q, -q^{14})$ (Entry 8.2.2). Then*

$$(3) \quad A^3 + B^3 = \frac{1}{q} \{ f^3(-q^6, -q^9) - f(-q, -q^4) a(q^5) \}.$$

Proof. Take $a = -q^6$, $b = -q^{-1}$ in (1), so again $ab = q^5$. Then $f(ab^2, a^2b) = f(-q^4, -q^{11}) = A$. Using $f(a, b) = af(a^{-1}, a^2b)$ [3, p. 34], $f(-q^{-1}, -q^{16}) = -q^{-1}f(-q, -q^{14})$ and $f(-q^{-1}, -q^6) = -q^{-1}f(-q, -q^4)$; hence $af^3(b, a^3b^2) = -q^6 f^3(-q^{-1}, -q^{16}) = q^3 f^3(-q, -q^{14}) = B^3$, $bf^3(a, a^2b^3) = -q^{-1} f^3(-q^6, -q^9)$ and $f(a, b) = -q^{-1} f(-q, -q^4)$. $\square$

Theorems 1–2 have exactly the shape of the recorded identities (8.2.9) and (8.2.13) — a cube of a level-15 theta function plus one further term — except that the second term contains the binary theta series $a(q^5)$ rather than a theta product. Since (1) is printed on the same page, immediately after the two “see note” remarks, it is natural to read Theorems 1–2 as the content of the lost notes; this reading is an interpretation, and we label it as such.

Literature. We found no statement of Theorems 1–2, nor any link between Entry 8.2.3 and the two “see note” remarks, in [2], in the chapter on circular summation of [5], in [6], in Chan–Liu–Ng [9] (full text), or in Ge–Luo [10]. Son’s paper [4] was checked only through its zbMATH review (Zbl 1062.11023), which lists the general theorem and the cases F_n , $n = 2, 3, 4, 5, 7$.

3. AN ELEMENTARY SECOND FORM

Theorem 3. *Let $A = f(-q^7, -q^8)$ and $B = qf(-q^2, -q^{13})$, as in Entry 8.2.1. Then*

$$(4) \quad A^3 - B^3 = \frac{f(-q^2, -q^3) f^3(-q, -q^4)}{f(-q^3, -q^{12})} + q^2 f^3(-q^3, -q^{12}) + 3AB \left\{ f(-q^{2/3}, -q) + q^{2/3} f(-q^3, -q^{12}) \right\},$$

where, by the Jacobi triple product,

$$AB = q f(-q^7, -q^8) f(-q^2, -q^{13}) = q (q^2, q^7, q^8, q^{13}; q^{15})_{\infty} (q^{15}; q^{15})_{\infty}^2.$$

Proof. By the polynomial identity $A^3 - B^3 = (A - B)^3 + 3AB(A - B)$, the claim is precisely (8.2.9) plus $3AB$ times (8.2.7), both of which are proved in [2, pp. 176–177]. The product form of AB is two applications of the Jacobi triple product $f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}$. $\square$

Theorem 4. *Let $A = f(-q^4, -q^{11})$ and $B = qf(-q, -q^{14})$, as in Entry 8.2.2. Then*

$$(5) \quad A^3 + B^3 = -\frac{1}{q} \left\{ \frac{f(-q, -q^4) f^3(-q^2, -q^3)}{f(-q^6, -q^9)} - f^3(-q^6, -q^9) \right\} + \frac{3AB}{q^{1/3}} \left\{ f(-q^{1/3}, -q^{4/3}) - f(-q^6, -q^9) \right\},$$

where

$$AB = q f(-q^4, -q^{11}) f(-q, -q^{14}) = q (q, q^4, q^{11}, q^{14}; q^{15})_{\infty} (q^{15}; q^{15})_{\infty}^2.$$

Proof. $A^3 + B^3 = (A + B)^3 - 3AB(A + B)$; substitute (8.2.13) and (8.2.11), proved in [2, p. 178]. $\square$

Both right-hand sides were verified coefficient-by-coefficient to 2000 terms (Section 6). We emphasize what is, and what is not, claimed: the two theorems are elementary consequences of Ramanujan's own four recorded identities, and in that sense they are certainly results Ramanujan possessed. Comparing (4) with (2) gives, as a by-product, $3AB(A - B) = f(-q^2, -q^3) \{a(q^5) - f^3(-q, -q^4)/f(-q^3, -q^{12})\}$ for the pair of Entry 8.2.1, and in the same way $3AB(A + B) = q^{-1} f(-q, -q^4) \{a(q^5) - f^3(-q^2, -q^3)/f(-q^6, -q^9)\}$ for the pair of Entry 8.2.2.

4. WHY A NOTE WAS NECESSARY: A NONEXISTENCE COMPUTATION

Each of the four cube identities Ramanujan recorded on page 54 is a one- or two-term combination of monomials in theta functions. It is natural to ask whether $A^3 - B^3$ (Entry 8.2.1) and $A^3 + B^3$ (Entry 8.2.2) also possess such a form, which Ramanujan would simply have recorded in the note. The evidence is that they do not.

4.1. Non-periodicity of the quadratic cofactors. Write $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$ and $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$. A power series with integer coefficients and constant term 1 has a unique representation $\prod_{n \geq 1} (1 - q^n)^{-c_n}$. If (c_n) is periodic with period M , the series equals

$\prod_{a=1}^M (q^a; q^M)_\infty^{-c_a}$, a generalized eta-quotient, i.e. a finite product of theta functions of the kind occurring throughout page 54 (Jacobi triple product); conversely every such product has periodic (c_n) . Computing (c_n) for $n \leq 380$ by the logarithmic-derivative recursion we find, in each entry, a sharp split:

Entry	combination	exponent sequence (c_n)
8.2.1	$A^2 - AB + B^2$ (factor of the recorded (8.2.8))	periodic mod 15
8.2.1	$A^2 + AB + B^2$ (factor of the lost $A^3 - B^3$)	not periodic
8.2.2	$A^2 + AB + B^2$ (factor of the recorded (8.2.12))	periodic mod 15
8.2.2	$A^2 - AB + B^2$ (factor of the lost $A^3 + B^3$)	not periodic

(“Not periodic” means: no period ≤ 60 holds for $n \leq 380$; the periodic cases were checked from $n = 1$.) Thus in each family the recorded cube identity is the one whose quadratic cofactor is a generalized eta-quotient, and the deferred one is the one whose cofactor is not. This already rules out any identity of the shape $A^3 \mp B^3 = (\text{linear factor known on p. 54}) \times (\text{generalized eta-quotient})$.

4.2. Systematic search over theta-monomial bases. We searched for representations

$$A^3 - B^3 = \sum_{i=1}^r c_i q^{s_i} M_i, \quad r \leq 2, \quad c_i \in \{\pm 1, \pm 2, \pm 3\}, \quad s_i \in \{0, \dots, 3\},$$

where each M_i is a monomial of total degree 3 with exponents in $\{-1, 1, 2, 3\}$ in at most three functions drawn from the 12-function basis

$$f(-q, -q^4), f(-q^2, -q^3), f(-q^3, -q^{12}), f(-q^6, -q^9), f(-q^5), \\ f(-q, -q^{14}), f(-q^2, -q^{13}), f(-q^4, -q^{11}), f(-q^7, -q^8), f(-q), f(-q^3), f(-q^{15})$$

— every theta function occurring in Entries 8.2.1–8.2.2 together with the relevant η -type functions (2344 monomials; hash-matching on the first 80 coefficients, surviving candidates confirmed on 260). The search was repeated in the $q^{1/3}$ -lattice (setting $q \mapsto q^3$ and adjoining the functions $f(-q^{a/3}, -q^{b/3})$ occurring in (8.2.7) and (8.2.11): 20 basis functions, 11800 monomials, shifts $s_i \in \{0, \dots, 9\}$), and with the three-term shape $c_1 q^{s_1} M_1 + c_2 q^{s_2} M_2 \pm 3q^s AB M_3$ suggested by Theorems 3–4.

Control of the instrument. Run on the recorded combinations, the same search re-finds Ramanujan’s identities exactly: for $(A - B)^3$ of Entry 8.2.1 it returns the two-term representation (8.2.9), and in the $q^{1/3}$ -lattice it returns (8.2.7) for $A - B$.

Result. For $A^3 - B^3$ (Entry 8.2.1) and $A^3 + B^3$ (Entry 8.2.2) the search returns nothing beyond trivial rewritings of the definitions. This is consistent with Theorems 1–2: the missing identities need the binary theta series $a(q^5)$, which lies outside the product basis. Within the declared space — which contains every identity actually appearing on page 54 — the two lost combinations have no one- or two-term product form.

We state the scope honestly: this is a finite search over an explicit basis and coefficient range, not a proof of nonexistence in general; a completeness argument via the theory of modular functions on $\Gamma_0(15)$ (the natural next step) is left open.

5. THE MIRROR SYMMETRY, AND A REMARK ON THE NOTE

The two entries are mirror images: under the exchange $f(-q^3, -q^{12}) \leftrightarrow f(-q^6, -q^9)$, $f(-q, -q^4) \leftrightarrow f(-q^2, -q^3)$, identity (8.2.8) corresponds to (8.2.12) and (8.2.9) to (8.2.13). In each family Ramanujan recorded the cube combination with the generalized eta-quotient cofactor and deferred the other to a note. The parallel structure of the two deferrals — the *same* obstruction on both sides of the mirror — is explained by Theorems 1–2: both notes are the same circular summation (1), with the same modulus $ab = q^5$ and the same factor $a(q^5)$, applied to the two mirror pairs. Whether Ramanujan intended precisely this, or a representation of a shape we have not searched, cannot of course be decided; we label this paragraph as interpretation, not theorem.

6. THE VERIFICATION SCRIPT

All twelve identities of this paper — (8.2.6)–(8.2.13), (4)–(5) and (2)–(3) — are certified by one self-contained Python script `verifier.py` (152 lines, no dependencies). It works in the variable $y = q^{1/3}$ with exact integer arithmetic; every identity involving a quotient is verified after cross-multiplication, so no division of power series occurs. Each identity is checked to 400 coefficients of q by default, and the run reported here used 2000 coefficients (6003 powers of y ; total runtime about one second). The script prints one MOS/FARQ (match/mismatch) line per identity and exits nonzero on any mismatch; the archived run prints 12/12 MOS. A second, independent script `sonli_tekshir.py` evaluates all identities as printed (with division and $q^{1/3}$) at five values of q with 50-digit arithmetic; the largest relative discrepancy is $2 \cdot 10^{-49}$. A third script `ildiz.py` checks the foundations independently: it compares every theta series used with its Jacobi triple product (3000 coefficients), verifies the circular summation (1) itself as an identity of two-variable integer power series in independent variables a, b (all monomials $a^i b^j$ with $i + j \leq 150$, where the brace is replaced by $a(ab)$), and checks $a^3 = b^3 + c^3$ in the division-free form $a(q)^3(q; q)_\infty^3(q^3; q^3)_\infty^3 = (q; q)_\infty^{12} + 27q(q^3; q^3)_\infty^{12}$ (3000 coefficients). Finally, `ildiz_gp.gp` re-implements the theta series, Theorems 1–2, both by-products and the Borwein identity from scratch in PARI/GP (1500 coefficients), independently of the Python code.

DATA AVAILABILITY

The verification script and the search scripts of Section 4 are archived at Zenodo: <https://doi.org/10.5281/zenodo.23030496> (all versions; version 1 is doi:10.5281/zenodo.23030497).

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