# Arc length of an elliptical helix via Ramanujan's second perimeter approximation, with explicit pitch thresholds

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**Abstract**

We observe that one turn of the elliptical helix $\mathbf{r}(t) = (a\cos t,\, b\sin t,\, ct)$ has exactly the length of the ellipse with semi-axes $\sqrt{a^2+c^2}$ and $\sqrt{b^2+c^2}$. Consequently Ramanujan's second approximation for the perimeter of an ellipse gives a closed-form turn length, and the pitch makes the approximation more accurate. We give explicit thresholds: if $c \geq 0.197254\,a$ the relative error is below $10^{-6}$ for every $b \in [0,a]$; if $c \geq 0.534032\,a$ it is below $10^{-9}$. The result gives integral-free wire and path lengths for elliptical coil springs, elliptical helical antennas and helical machining paths. Version 2 adds an exact formula for coils with stepped pitch and part turns, and closed-form estimates for progressive-pitch and tapered elliptical coils.

## 1 The reduction

Let $a \geq b \geq 0$, $a > 0$, $c > 0$, and write $A = \sqrt{a^2+c^2}$, $B = \sqrt{b^2+c^2}$.

**Lemma 1.** *The length of one turn of $\mathbf{r}(t) = (a\cos t, b\sin t, ct)$, $0 \leq t \leq 2\pi$, equals the perimeter $P(A,B)$ of an ellipse with semi-axes $A$ and $B$.*

*Proof.* $|\mathbf{r}'(t)|^2 = a^2\sin^2 t + b^2\cos^2 t + c^2 = (a^2+c^2)\sin^2 t + (b^2+c^2)\cos^2 t = A^2\sin^2 t + B^2\cos^2 t$, which is the squared speed of $(A\cos t, B\sin t)$. Integrating over $[0,2\pi]$ gives $P(A,B) = 4A\,E(1 - B^2/A^2)$. $\square$

## 2 Error of Ramanujan's approximation

Ramanujan's second approximation [1] is

$$R(A,B) = \pi(A+B)\left(1 + \frac{3h}{10+\sqrt{4-3h}}\right), \qquad h = \left(\frac{A-B}{A+B}\right)^2.$$

Since $P/(\pi(A+B)) = \sum_{n\geq 0} \binom{1/2}{n}^2 h^n$, the difference $\big(P-R\big)/(\pi(A+B))$ has vanishing coefficients up to $h^4$ and leading term $\frac{3}{2^{17}}h^5$; we verified symbolically that its coefficients are positive through $h^{25}$, and on a 2001-point grid (50-digit arithmetic) that the relative error $\varepsilon(h) = (P-R)/P$ is increasing. This agrees with the error analyses in [2, 3].

Because $b \geq 0$, $B/A \geq k(\rho) := \rho/\sqrt{1+\rho^2}$ with $\rho = c/a$, hence $h \leq h_{\max}(\rho) = \big((1-k)/(1+k)\big)^2$.

**Theorem 1.** *For the helix above, the relative error of $R(A,B)$ as an approximation of the turn length satisfies $\varepsilon \leq \varepsilon\big(h_{\max}(c/a)\big)$, with equality at $b=0$. In particular*

$$\frac{c}{a} \;\geq\; 0.197254 \quad\Longrightarrow\quad \varepsilon < 10^{-6} \qquad \textit{for every } b \in [0,a].$$

At $c = 0.2a$ one gets $k = 0.196116\ldots$, $h_{\max} = 0.451689\ldots$ and worst-case error $9.3758 \times 10^{-7}$.

| worst-case relative error $\leq$ | required pitch $c/a$ $\geq$ |
|---|---|
| $10^{-4}$ | 0.0314394 |
| $10^{-5}$ | 0.1057625 |
| $10^{-6}$ | 0.197254 |
| $10^{-7}$ | 0.300449 |
| $10^{-8}$ | 0.4129297 |
| $10^{-9}$ | 0.5340313 |

Table 1: Pitch thresholds (worst case over $b \in [0, a]$). For $c/a \to 0$ the worst case tends to the classical $4.02 \times 10^{-4}$ of the flat ellipse.

## 3 Numerical check

For 300 random helices with $a = 1$, $b \in (0, 1)$, $c \in (0.2, 3)$ the turn length was computed by direct adaptive quadrature of $|\mathbf{r}'(t)|$ and compared with $R(A, B)$; the largest relative deviation was $8.17 \times 10^{-8}$, consistent with the theorem.

## 4 Applications

The formula gives the wire length per turn of elliptical coil springs and elliptical helical antennas (studied since [4]) and the path length of helical machining or in-pipe robot trajectories, without evaluating an elliptic integral, at part-per-million accuracy whenever $c \geq 0.2a$.

## 5 Extensions (added in version 2)

Throughout, $a, b$ are the semi-axes of the winding, $c_i = p_i/2\pi$, $A_i = \sqrt{a^2 + c_i^2}$, $B_i = \sqrt{b^2 + c_i^2}$, and all lengths were compared with direct quadrature of $|\mathbf{r}'(t)|$ at 30 digits.

**Stepped pitch and part turns (exact).** A coil made of sections of $n_i$ turns at constant pitch $p_i$ has length $\sum_i n_i P(A_i, B_i)$ when every $n_i$ is an integer. A part turn is an incomplete elliptic integral: with $m = 1 - B^2/A^2$,

$$\int_0^{\varphi} \sqrt{A^2 \sin^2 t + B^2 \cos^2 t}\, dt = A\big[E(m) - E(\tfrac{\pi}{2} - \varphi, m)\big] \qquad (0 \leq \varphi \leq \tfrac{\pi}{2}),$$

and each further quarter turn adds $A\,E(m)$; if the part turn ends $\varphi'$ radians into an odd-numbered quarter (one starting at $t = \pi/2 + k\pi$), the last piece is $A\,E(\varphi', m)$ instead. For the sections $(1.5, 3)$, $(6, 15)$, $(1.25, 4)$ (turns, pitch in mm) on $a = 10$, $b = 6$ this gives $460.37961606241$ mm, equal to the quadrature value to 30 digits. Replacing $P$ by Ramanujan's $R$ gives integral-free lengths with the error of Theorem 1.

**Progressive pitch.** For pitch rising linearly from $p_0$ to $p_1$ over $N$ turns, Simpson's rule applied turn by turn to $R\big(\sqrt{a^2 + c^2}, \sqrt{b^2 + c^2}\big)$, with $c$ at the start, middle and end of each turn, gave relative errors $1.2 \times 10^{-6}$ ($p$: $4 \to 16$ mm, $N = 8$), $5.8 \times 10^{-6}$ ($5 \to 30$, $N = 6$) and $2.0 \times 10^{-8}$ ($10 \to 12$, $N = 10$).

**Tapered coils.** For a round conical helix $\big(r(t)\cos t,\ r(t)\sin t,\ ct\big)$ with $r(t) = r_0 + \alpha t$ the length is the classical closed form $\big[r\sqrt{r^2 + k^2} + k^2 \operatorname{arsinh}(r/k)\big]/(2\alpha)$ between $r_0$ and $r_1$, $k^2 = \alpha^2 + c^2$. For an elliptical taper $a(t) = a_0 + \alpha t$, $b(t) = b_0 + \beta t$, summing $R\big(\sqrt{a^2 + \beta^2 + c^2}, \sqrt{b^2 + \alpha^2 + c^2}\big)$ at mid-turn values gave relative errors $8.3 \times 10^{-6}$, $1.9 \times 10^{-6}$ and $1.0 \times 10^{-7}$ on three test coils.

**Changes in version 2.** Section 5 is new. Two threshold entries of Table 1 were rounded up so that each stated threshold is sufficient (version 1 gave 0.300448 and 0.4129296).

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# References

[1] S. Ramanujan, Modular equations and approximations to $\pi$, *Quart. J. Math.* 45 (1914) 350–372.

[2] M. B. Villarino, Ramanujan's perimeter of an ellipse, arXiv:math/0506384 (2005).

[3] J.-F. Tian, Z.-H. Yang, Absolute monotonicity of the accuracy of Ramanujan approximations for perimeter of an ellipse, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. (RACSAM)* 117(3) (2023), doi:10.1007/s13398-023-01465-z.

[4] J. Wong, S. Loh, Radiation field of an elliptical helical antenna, *IRE Trans. Antennas Propag.* 7(1) (1959) 46–52, doi:10.1109/TAP.1959.1144634.